Linear Transformation

$\phi : \mathbb{R}^n \to \mathbb{R}^n$

$\phi(\vec{v}) = A\vec{v}$ for some A Ex $proj_V : \mathbb{R}^n \to \mathbb{R}^n$ ST "A" $= A(A^T A^{-1}) A^T$

$\phi_1 \circ \phi_2 = A_1 A_2$

$\phi^{-1} = A^{-1}$

$B = V^{-1} A V$, ST B is the change of bais matrix, V = vectors that you want your basis in the space of.

$$\boxed{\underset{\text{out}}{B} = W^{-1} A \underset{\text{input}}{V}}$$

Determinant

$\det A \to$ factor by which lin transformation ϕ scales n-dim volumes of regions in $\mathbb{R}^n$.

Ex → some things flip area (-1)
some things reduce dim (0)

$\det AB = \det(A)\det(B)$

→ If A is square, $\det A = \pm$ (products of pivots of REF(A))

→ If rows/cols A are linearly dependent, then $\det A = 0$

→ $\det A = \det A^T$

$$\det \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \boxed{a_1+b_1 \quad a_2+b_2 \quad \cdots \quad a_n+b_n} \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} = \det \begin{bmatrix} - & - & - \\ a_1 & a_2 \cdots & a_n \\ - & - & - \end{bmatrix} + \det \begin{bmatrix} - & - & - \\ b_1 & b_2 \cdots & b_n \\ - & - & - \end{bmatrix}$$

$\det A = a_{1j} C_{1j} + \dots a_{nj} \cdot C_{nj}$

↳ $C_{ij} = (-1)^{i+j} \det M_{ij}$ → remove row i, col j from A

Back to Inverses

$A^{-1}_{ij} = \dfrac{C_{ji}}{\det A}$

For some $A\vec{v} = \vec{b}$, $\vec{v} = A^{-1}\vec{b}$, so $v_i = \dfrac{\det B_i}{\det A}$, $B_i = A$ w/ ith col $= \vec{b}$

$$\boxed{A^{-1} = \frac{1}{\det(A)} X, \; X = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}}$$

$\det(A-\lambda I)=0$ for $Av=\lambda v$

$p(\lambda)=\det(A-\lambda I)$

$tr = \sum \text{diags} \ldots = \sum_{\lambda=\text{eigenvalue}} \lambda$

$v \perp w \Rightarrow$ ~~[scribble]~~ $v \cdot w = v^T w = 0$ <u>Def orthogonality</u>: $\|v+w\|^2 = \|v\|^2 + \|w\|^2$

<u>orthogonal subspaces</u> $\rightarrow V \perp W$, $\forall\, v \in V, w \in W$

$C(A) \perp N(A^T)$ $\quad N(A) \perp C(A^T)$] Fundamental Thm of Lin Alg

<u>orthogonal complement</u>: $V^{\perp} = \{ w \in \mathbb{R}^n \text{ ST } v \perp w \text{ for all } v \in V \}$

Rule: 2 complementary subspaces $V, W \rightarrow$ any $a \in \mathbb{R}^n$ can be represented as $a = v + w$, $v \in V, w \in W$

<u>PROJECTION</u>

$\vec{p} = \text{proj}_V \vec{b} \in V \quad \text{ST } (b - p) \perp V$

↓

projection of $\vec{b}$ onto V. Best guess of $\vec{b}$ w/ restrictions set in V.

$$\text{proj}_{\vec{a}} \vec{b} = \left(\frac{a^T b}{a^T a}\right)\vec{a} \qquad P_a = \frac{a a^T}{a^T a} \rightarrow \text{projection matrix onto line } \vec{a}$$

$$\text{proj}_{C(A)} \vec{b} = A(A^T A)^{-1} A^T \vec{b}$$ → projection of $\vec{b}$ onto $C(A)$ is space spanned by cols in A.

$$P_{C(A)} = A(A^T A)^{-1} A^T$$

Least Squares $\rightarrow$ $\vec{e} = \vec{b} - A\vec{v}$ $\qquad \|\vec{b} - A\vec{v}\| \geq \|\vec{b} - \vec{p}\|$

↳ orthogonal projection of $\vec{b}$ onto $C(A)$

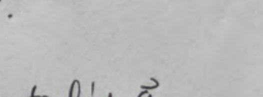
$\vec{e}$: error; $\vec{b}$: actual vector; $A\vec{v}$: ↳ best guess

① What is $\vec{p}$ on $C(A)$?

② What is $\vec{v}$ ST $A\vec{v} = \vec{p}$

$$\vec{v} = (A^T A)^{-1} A^T b$$

or $0 = -A^T b + A^T A v$



collection of $q_1 \ldots q_n$ are orthogonal if $q_i \perp q_j \; \forall i, j$

orthonormal " " " & $\|q_i\| = 1$

$$Q^T Q = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ 0 & & d_3 \ddots & \\ & & & d_n \end{bmatrix}, \text{ ST } d_i = \|q_i\|^2$$

$\boxed{Q^T Q = I_n}$ if orthonormal

Matrix → orthogonal if $Q^T = Q^{-1}$. $(Q\vec{v})^T(Q\vec{w}) = \vec{v}^T\vec{w}$, $\forall \vec{v}, \vec{w}$

$$\left[q_1 \middle| q_2 \middle| \cdots \middle| q_n \right]$$

*** Imagine you want to project onto some space $C(A)$

$C(A) \to$ orthogonal vectors representing space.

$\hookrightarrow Q$.

Then $P_{C(A)} = P_{C(Q)} = QQ^T$.

How to get Q? $C(A) = v_1, v_2, \ldots$

A

$v_1 \rightsquigarrow q_1 = \frac{v_1}{\|v_1\|}$ $v_2 \rightsquigarrow w_2 = v_2 - \text{proj}_{q_1} v_2$ $v_3 \rightsquigarrow w_3 = v_3 - \text{proj}_{q_1} v_3 - \text{proj}_{q_2} v_3$

$w_2 \rightsquigarrow q_2 = \frac{w_2}{\|w_2\|}$ $w_3 \rightsquigarrow q_3 = \frac{w_3}{\|w_3\|}$

$A \rightsquigarrow AD_1^{(\frac{1}{\|v_1\|})}$ $A \rightsquigarrow AE_{12}^{\lambda_{12}} D_2^{1/\|w_2\|}$

ST λ_{12} = some constant

$Q = A\ DEDEEE\ldots$

~~QE AR~~

$\boxed{A = QR}$ R is upper Δ